



Variable-order Implicit Fractional Differential Equations based on the Kuratowski MNC Technique

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Abstract

In this manuscript, we examine the existence and the stability of solutions to the boundary value problem of Riemann-Liouville fractional differential equations of variable order. The obtained new results are based on the fixed point theorem of Darbo and Kuratowski's metric of non-compactness (MNC) with the help of piece-wise constant functions. In addition, the derived fundamental results are proven suitable because they satisfy the Ulam-Hyers Rassias stability sufficient conditions. Several numerical examples were discussed too to demonstrate the reasonableness and effectiveness of the observed results.

Keywords: fractional differential equations of variable order; boundary value problem; Darbo's fixed point theorem; measure of non-compactness; Ulam Hyers Rassias stability; Green function.

1 Introduction

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number powers or complex number powers of the differentiation and integration operators [9]. In the last thirty years, fractional calculus has contributed to a multitude of significant discoveries in pure and applied mathematics and various domains, such as chemistry [9], physics [16], biology [1], control theory [17], economics [12], biophysics [10], signal [24] and image processing, etc [8, 7].

The study of fractional differential equations has attracted the attention of several researchers, resulting in the publication of numerous papers that address various in this field. For example, Baleanu et al. [2] used the Caputo and Fabrizio fractional derivative to express the model of HIV and solve the equation. Ahmad et al. [17] presented the existence of solutions for nonlinear neutral stochastic fractional differential systems. Shah et al. [23] obtained sufficient conditions for the existence of solutions to the coupled system of nonlinear boundary value problems. Tuan et al. [25] presented a mathematical model for the transmission of COVID-19 by the Caputo fractional-order derivative.

The existence of solutions to variable-order problems is rarely discussed in the literature. Souid et al. [6, 18] presented the existence, uniqueness and stability of solutions to many different problems (implicit, resonance). Stability theory is very significant, as each practicable control system is designed to be stable. The analysis of solution stability has captivated many researchers due to its promising potential (see [14, 22]).

Motivated by the previous research, we deal with the following boundary value problems (BVP):

$$\begin{cases} D_{0+}^{\varphi(s)} \xi(s) + h(s, \xi(s), D_{0+}^{\varphi(s)} \xi(s)) = 0, & s \in \Upsilon := [0, K], \\ \xi(0) = 0, \quad \xi(K) = 0. \end{cases} \tag{1}$$

For $1 < \varphi(s) \leq 2$, the function $h : \Upsilon \times \Lambda \times \Lambda \rightarrow \Lambda$ is continuous, Λ is a real (or complex) Banach space and $D_{0+}^{\varphi(s)}$ is the Riemann-Liouville fractional derivative of order variable $\varphi(s)$.

2 Auxiliary Notions

Let the Banach space $C(\Upsilon, \Lambda)$ the set of the real-valued continuous functions defined on the interval Υ , equipped with the usual norm:

$$\|\varkappa\| = \text{Sup}\{\|\varkappa(s)\| : s \in \Upsilon\}.$$

Definition 2.1. ([20, 21]) For $-\infty < \tau_1 < \tau_2 < +\infty$ and $\varphi(s) : [\tau_1, \tau_2] \rightarrow (0, +\infty)$, the Riemann Liouville fractional integral FI(RL) of variable order $\varphi(s)$ for function v is defined by:

$$I_{\tau_1+}^{\varphi(s)} v(s) = \int_{\tau_1}^s \frac{(s - \alpha)^{\varphi(s)-1}}{\Gamma(\varphi(s))} v(\alpha) d\alpha, \quad s > \tau_1, \tag{2}$$

where Γ is the gamma function.

Definition 2.2. ([20, 21, 26]) For $-\infty < \tau_1 < \tau_2 < +\infty$ and $\varphi(s) : [\tau_1, \tau_2] \rightarrow (n - 1, n)$, $n \in \mathbb{N}$, the Riemann Liouville fractional derivative $FD(RL)$ of variable-order $\varphi(s)$ for function v is defined by:

$$D_{\tau_1^+}^{\varphi(s)} v(s) = \left(\frac{d}{ds}\right)^n I_{\tau_1^+}^{n-\varphi(s)} v(s) = \left(\frac{d}{ds}\right)^n \int_{\tau_1}^s \frac{(s-\alpha)^{n-\varphi(s)-1}}{\Gamma(n-\varphi(s))} v(\alpha) d\alpha, \quad s > \tau_1. \tag{3}$$

As expected, FI(RL) and FD(RL) correspond with the usual Riemann Liouville fractional integral and Riemann Liouville derivative: [20, 15]. Consider the following essential observations.

Lemma 2.1. ([15]) Suppose $\varrho, \varpi > 0, \tau_1 > 0$ and $v \in L(\tau_1, \tau_2), D_{\tau_1^+}^\varrho v \in L(\tau_1, \tau_2)$. Then, the differential equation

$$D_{\tau_1^+}^\varrho v = 0,$$

has solution

$$v(s) = \sum_{\iota=1}^{\iota=n} \eta_\iota (s - \tau_1)^{\varrho-\iota},$$

and

$$I_{\tau_1^+}^\varrho D_{\tau_1^+}^\varrho v(s) = v(s) + \sum_{\iota=1}^{\iota=n} \eta_\iota (s - \tau_1)^{\varrho-\iota},$$

where $\eta_\iota \in \mathbb{R}, \iota = 1, 2, \dots, n$ with $n - 1 < \varrho \leq n$. Furthermore,

$$D_{\tau_1^+}^\varrho I_{\tau_1^+}^\varrho v(s) = v(s),$$

and

$$I_{\tau_1^+}^\varrho I_{\tau_1^+}^\varpi v(s) = I_{\tau_1^+}^\varpi I_{\tau_1^+}^\varrho v(s) = I_{\tau_1^+}^{\varrho+\varpi} v(s).$$

Remark([29, 27, 32]) Note that the general functions $\varrho(s)$ and $\varpi(s)$ do not satisfy the semigroup condition, i.e.,

$$I_{\tau_1^+}^{\varrho(s)} I_{\tau_1^+}^{\varpi(s)} v(s) \neq I_{\tau_1^+}^{\varrho(s)+\varpi(s)} v(s).$$

Example 2.1: Let

$$\varrho(s) = \frac{s}{2}, \quad s \in [0, 3], \quad \varpi(s) = \begin{cases} 1, & s \in [0, 1], \\ 3, & s \in [1, 3], \end{cases} \quad v(s) = 4, \quad s \in [0, 3].$$

Thus,

$$\begin{aligned} I_{0^+}^{\varrho(s)} I_{0^+}^{\varpi(s)} v(s) &= \int_0^s \frac{(s-\alpha)^{\varrho(s)-1}}{\Gamma(\varrho(s))} \int_0^\alpha \frac{(\alpha-\sigma)^{\varpi(\alpha)-1}}{\Gamma(\varpi(\alpha))} v(\sigma) d\sigma d\alpha, \\ &= \int_0^s \frac{(s-\alpha)^{\varrho(s)-1}}{\Gamma(\varrho(s))} \left[\int_0^1 \frac{(\alpha-\sigma)^0}{\Gamma(1)} 4 d\sigma + \int_1^\alpha \frac{(\alpha-\sigma)^2}{\Gamma(3)} 4 d\sigma \right] d\alpha, \\ &= \int_0^s \frac{(s-\alpha)^{\varrho(s)-1}}{\Gamma(\varrho(s))} \left[4 + \frac{2}{3}(\alpha-1)^3 \right] d\alpha, \end{aligned}$$

and

$$I_{0+}^{\varrho(s)+\varpi(s)}v(s) = \int_0^s \frac{(s - \alpha)^{\varrho(s)+\varpi(s)-1}}{\Gamma(\varrho(s) + \varpi(s))}v(\alpha) d\alpha.$$

Consequently, we get

$$\begin{aligned} I_{0+}^{\varrho(s)}I_{0+}^{\varpi(s)}v(s)|_{s=2} &= \int_0^2 \frac{(2 - \alpha)^0}{\Gamma(1)} \left[4 + \frac{2}{3}(\alpha - 1)^3 \right] d\alpha, \\ &= \int_0^2 \left(\frac{2}{3}\alpha^3 - 2\alpha^2 + 2\alpha + \frac{10}{3} \right) d\alpha, \\ &= 8. \end{aligned}$$

$$\begin{aligned} I_{0+}^{\varrho(s)+\varpi(s)}v(s)|_{s=2} &= \int_0^2 \frac{(2 - \alpha)^{\varrho(s)+\varpi(s)-1}}{\Gamma(\varrho(s) + \varpi(s))}v(\alpha) d\alpha, \\ &= \int_0^1 \frac{(2 - \alpha)^1}{\Gamma(2)}4 d\alpha + \int_1^2 \frac{(2 - \alpha)^3}{\Gamma(4)}4 d\alpha, \\ &= 4 \int_0^1 (-\alpha + 2) d\alpha + \frac{2}{3} \int_1^2 (-\alpha^3 + 6\alpha^2 - 12\alpha + 8) d\alpha, \\ &= 6 + \frac{1}{6} = \frac{37}{6}. \end{aligned}$$

Therefore, we obtain

$$I_{0+}^{\varrho(s)}I_{0+}^{\varpi(s)}v(s)|_{s=2} \neq I_{0+}^{\varrho(s)+\varpi(s)}v(s)|_{s=2}.$$

Lemma 2.2. ([31]) Let $\varphi \in C(\Upsilon, (1, 2])$ and $0 \leq \delta \leq \min_{s \in \Upsilon} |\varphi(s)|$, then for any $v \in C_\delta(\Upsilon, \Lambda)$ where $C_\delta(\Upsilon, \Lambda) = \{v(s) \in C(\Upsilon, \Lambda), s^\delta v(s) \in C(\Upsilon, \Lambda)\}$, the integral $I_{0+}^{\varphi(s)}v(s)$ exists for any $s \in \Upsilon$.

Lemma 2.3. ([31]) If $\varphi \in C(\Upsilon, (1, 2])$, then $I_{0+}^{\varphi(s)}v(s) \in C(\Upsilon, \Lambda)$ for any $v \in C(\Upsilon, \Lambda)$.

Definition 2.3. ([30, 13, 28]) Let $I \subset \mathbb{R}$;

1. The generalised interval I is either an interval or $\{a\}$ or \emptyset .
2. A partition of I is a finite set \mathcal{P} , such that, for any x in I lies in exactly one of the generalised intervals E in \mathcal{P} .
3. Let \mathcal{P} be a partition of I , we say that the function $h : I \rightarrow \mathbb{R}$ is piecewise constant with respect to \mathcal{P} if for each E in \mathcal{P} , h admits constant values on E .

2.1 The technique of measures of non-compactness

Definition 2.4. ([3]) Let Ω_Λ the bounded subsets of a Banach space Λ . The (MNK) is a mapping $\vartheta : \Omega_\Lambda \rightarrow [0, \infty]$ that follows the format below,

$$\vartheta(\Delta) = \inf\{\epsilon > 0 : \Delta(\in \Omega_\Lambda) \subseteq \cup_{i=1}^n \Delta_i, \text{diam}(\Delta_i) \leq \epsilon\},$$

where

$$\text{diam}(\Delta_i) = \sup\{\|x - y\| : x, y \in \Delta_i\}.$$

Proposition 2.1. ([3, 4]) Let $\Delta, \Delta_1, \Delta_2$ be a bounded subsets of Λ , then:

1. Δ is relatively compact $\iff \vartheta(\Delta) = 0$.
2. $\vartheta(\emptyset) = 0$.
3. $\vartheta(\Delta) = \vartheta(\overline{\Delta}) = \vartheta(\text{conv}\Delta)$.
4. $\Delta_1 \subset \Delta_2 \implies \vartheta(\Delta_1) \leq \vartheta(\Delta_2)$.
5. $\vartheta(\Delta_1 + \Delta_2) \leq \vartheta(\Delta_1) + \vartheta(\Delta_2)$.
6. $\vartheta(\Pi\Delta) = |\Pi|\vartheta(\Delta), \Pi \in R$.
7. $\vartheta(\Delta_1 \cup \Delta_2) = \text{Max}\{\vartheta(\Delta_i), i = 1, 2\}$.
8. $\vartheta(\Delta_1 \cap \Delta_2) = \text{Min}\{\vartheta(\Delta_i), i = 1, 2\}$.
9. $\vartheta(\Delta + x_0) = \vartheta(\Delta)$ for any $x_0 \in \Lambda$.

Lemma 2.4. ([11]) Let Λ be a Banach space. If χ is an bounded and equicontinuous subset of the space $C(\Upsilon, \Lambda)$ of continuous functions, then;

(i) $\vartheta(\chi(s))$ is a continuous function for $s \in \Upsilon$, and

$$\widehat{\vartheta}(\chi) = \sup_{s \in \Upsilon} \vartheta(\chi(s)).$$

(ii) $\vartheta\left(\int_0^K x(\theta) d\theta : x \in \chi\right) \leq \int_0^K \vartheta(\chi(\theta)) d\theta,$

where

$$\chi(\alpha) = \{x(\alpha) : x \in \chi\}, \alpha \in \Upsilon.$$

Theorem 2.1. (DFPTh) [3] Let Υ be a nonempty, closed, bounded, and convex subset of a Banach space Λ , and assume that $F : \Upsilon \rightarrow \Upsilon$ is a continuous operator fulfilling:

$$\vartheta(F(S)) \leq l\vartheta(S), \quad \text{for any } S(\neq \emptyset) \subset \Upsilon, \quad l \in [0, 1),$$

i.e., F is l -set contractions. Therefore, F has at least one fixed point in Υ .

Definition 2.5. ([19]) The equation (1) is Ulam-Hyers stable if $c_h > 0$ exists, such that for any $\epsilon > 0$ and for every solution $z \in C(\Upsilon, \Lambda)$, the following inequality holds.

$$\left\| D_{0+}^{\varphi(s)} z(s) + h\left(s, z(s), D_{0+}^{\varphi(s)} z(s)\right) \right\| \leq \epsilon, \quad s \in \Upsilon.$$

There is a solution $\xi \in C(\Upsilon, \Lambda)$ of equation (1) with:

$$\|z(s) - \xi(s)\| \leq c_h \epsilon, \quad s \in \Upsilon.$$

Definition 2.6. ([19]) The equation (1) is (HUR) stable with respect to ψ if $c_h > 0$ exists such that for each $\epsilon > 0$ and every solution $z \in C(\Upsilon, \Lambda)$ of the following inequality:

$$\left\| D_{0+}^{\varphi(s)} z(s) + h\left(s, z(s), D_{0+}^{\varphi(s)} z(s)\right) \right\| \leq \epsilon\psi(s), \quad s \in \Upsilon,$$

(1) has a solution $\xi \in C(\Upsilon, \Lambda)$ satisfying

$$\|z(s) - \xi(s)\| \leq c_h \epsilon \psi(s), \quad s \in \Upsilon.$$

3 The Existence of Solutions for Boundary Value Problem

Let's introduce these presumptions:

(H1) Let $n \in \mathbb{N}$ and $\beta = \{\Upsilon_1 := [0, K_1], \Upsilon_2 := (K_1, K_2], \Upsilon_3 := (K_2, K_3], \dots, \Upsilon_n := (K_{n-1}, K]\}$ represent a partition of the interval Υ , and let $\varphi(t) : \Upsilon \rightarrow (1, 2]$ be a piecewise constant function with respect to β , i.e.,

$$\varphi(s) = \sum_{\iota=1}^n \varphi_{\iota} I_{\iota}(s) = \begin{cases} \varphi_1, & \text{if } s \in \Upsilon_1, \\ \varphi_2, & \text{if } s \in \Upsilon_2, \\ \vdots & \\ \varphi_n, & \text{if } s \in \Upsilon_n, \end{cases}$$

where $1 < \varphi_{\iota} \leq 2$ are constants and I_{ι} indicates the interval for $\Upsilon_{\iota} := (K_{\iota-1}, K_{\iota}]$, $\iota = 1, 2, \dots, n$, (with $K_0 = 0, K_n = K$) such that,

$$I_{\iota}(s) = \begin{cases} 1, & \text{for } s \in \Upsilon_{\iota}, \\ 0, & \text{for elsewhere.} \end{cases}$$

(H2) Let $s^{\delta} h : \Upsilon \times \Lambda \times \Lambda \rightarrow \Lambda$ be a continuous function ($0 \leq \delta \leq \min_{s \in \Upsilon} |(\varphi(s))|$), there exist a constants $\gamma_1, \gamma_2 > 0$ such that

$$s^{\delta} \left\| h(s, \pi_1, \varsigma_1) - h(s, \pi_2, \varsigma_2) \right\| \leq \gamma_1 \|\pi_1 - \pi_2\| + \gamma_2 \|\varsigma_1 - \varsigma_2\|,$$

for any $\pi_1, \pi_2, \varsigma_1, \varsigma_2 \in \Lambda$.

Remark 3.1. According to the previous observation of [5], we can easily show that the condition (H2) and the following inequality

$$\vartheta \left(t^{\delta} \left\| h(s, D_1, D_2) \right\| \right) \leq \gamma_1 \vartheta(D_1) + \gamma_2 \vartheta(D_2),$$

are equivalent for any bounded sets $D_1, D_2 \subset \Lambda$ and every $s \in \Upsilon$.

Furthermore, let's define $\varphi : \Upsilon \rightarrow \Lambda$ for a given set of functions χ indicated by,

$$\chi(s) = \left\{ x(s), \quad x \in \chi \right\}, \quad s \in \Upsilon,$$

and

$$\chi(\Upsilon) = \left\{ \chi(s) : x \in \chi, \quad s \in \Upsilon \right\}.$$

Now, using the concepts of (MNK) and (DFPTh), we can show that there is a solution to (BVP)(1).

The notation $\Xi_{\iota} = C(\Upsilon_{\iota}, \Lambda)$ signifies the Banach space of continuous functions $\xi : \Upsilon_{\iota} \rightarrow \Lambda$ for each $\iota \in \{1, 2, \dots, n\}$, equipped with,

$$\|\xi\|_{\Xi_{\iota}} = \sup_{s \in \Upsilon_{\iota}} \|\xi(s)\|.$$

We begin by analysing BVP as defined in (1).

Because of (3), we can express the equation of the BVP(1) as follows:

$$\frac{d^2}{ds^2} \int_0^s \frac{(s - \alpha)^{1-\varphi(s)}}{\Gamma(2 - \varphi(s))} \xi(\alpha) d\alpha + h\left(s, \xi(s), D_{0+}^{\varphi(s)} \xi(s)\right) = 0, \quad s \in \Upsilon. \tag{4}$$

The equation (4) in Υ_ι , where $\iota = 1, 2, \dots, n$ may be presented as follows while accounting for (H1),

$$\frac{d^2}{ds^2} \left(\int_0^{K_1} \frac{(s - \alpha)^{1-\varphi_1}}{\Gamma(2 - \varphi_1)} \xi(\alpha) d\alpha + \dots + \int_{K_{\iota-1}}^s \frac{(s - \alpha)^{1-\varphi_\iota}}{\Gamma(2 - \varphi_\iota)} \xi(\alpha) d\alpha \right) + h\left(s, \xi(s), D_{0+}^{\varphi_\iota} \xi(s)\right) = 0, \quad s \in \Upsilon_\iota. \tag{5}$$

Definition 3.1. BVP (1) has a solution, if there exists functions $\xi_\iota, \iota = 1, 2, \dots, n$, such that, $\xi_\iota \in C([0, K_\iota], \Lambda)$ satisfying equation (5) and $\xi_\iota(0) = 0 = \xi_\iota(K_\iota)$.

If we consider $\xi(s) \equiv 0$ for $0 \leq s \leq K_{\iota-1}$, then (5) is expressed as,

$$D_{K_{\iota-1}^+}^{\varphi_\iota} \xi(s) + h\left(s, \xi(s), D_{K_{\iota-1}^+}^{\varphi_\iota} \xi(s)\right) = 0, \quad s \in \Upsilon_\iota.$$

We will address the following BVPs:

$$\begin{cases} D_{K_{\iota-1}^+}^{\varphi_\iota} \xi(s) + h\left(s, \xi(s), D_{K_{\iota-1}^+}^{\varphi_\iota} \xi(s)\right) = 0, & s \in \Upsilon_\iota, \\ \xi(K_{\iota-1}) = 0, \quad \xi(K_\iota) = 0. \end{cases} \tag{6}$$

Lemma 3.1. The solution ξ of (6) is written as the following integral equation

$$\xi(s) = \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \alpha) h\left(\alpha, \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) \xi(\sigma) d\sigma, \xi(\alpha)\right) d\alpha, \tag{7}$$

where $D_{K_{\iota-1}^+}^{\varphi_\iota} \xi(s) = \xi(s)$, $G_\iota(s, \alpha)$ is the Green’s function, defined as follows:

$$G_\iota(s, \alpha) = \begin{cases} \frac{1}{\Gamma(\varphi_\iota)} \left[(K_\iota - K_{\iota-1})^{1-\varphi_\iota} (s - K_{\iota-1})^{\varphi_\iota-1} (K_\iota - \alpha)^{\varphi_\iota-1} - (s - \alpha)^{\varphi_\iota-1} \right], & K_{\iota-1} \leq \alpha \leq s \leq K_\iota, \\ \frac{1}{\Gamma(\varphi_\iota)} (K_\iota - K_{\iota-1})^{1-\varphi_\iota} (s - K_{\iota-1})^{\varphi_\iota-1} (K_\iota - \alpha)^{\varphi_\iota-1}, & K_{\iota-1} \leq s \leq \alpha \leq K_\iota, \end{cases}$$

where $\iota = 1, 2, \dots, n$.

Proof. We assume that $\xi \in \Xi_\iota$ is the solution to the problem (6), and we take $D_{K_{\iota-1}^+}^{\varphi_\iota} \xi(s) = \xi(s)$. Applying the operator $I_{K_{\iota-1}^+}^{\varphi_\iota}$ to both sides of (6), we find (see Lemma(2.1)),

$$\xi(s) = \omega_1 (s - K_{\iota-1})^{\varphi_\iota-1} + \omega_2 (s - K_{\iota-1})^{\varphi_\iota-2} - I_{K_{\iota-1}^+}^{\varphi_\iota} \xi(s), \quad s \in \Upsilon_\iota.$$

Due to the assumption of function h together with $\xi(K_{\iota-1}) = 0$, we conclude that $\omega_2 = 0$. □

Let $\xi(s)$ satisfies $\xi(K_l) = 0$. We can observe that

$$\omega_1 = (K_l - K_{l-1})^{1-\varphi_l} I_{K_{l-1}^+}^{\varphi_l} \xi(K_l).$$

Then, we observe

$$\xi(s) = (K_l - K_{l-1})^{1-\varphi_l} (s - K_{l-1})^{\varphi_l-1} I_{K_{l-1}^+}^{\varphi_l} \xi(K_l) - I_{K_{l-1}^+}^{\varphi_l} \xi(s), \quad s \in \Upsilon_l.$$

Then the solution to problem (6) is given by:

$$\begin{aligned} \xi(s) &= (K_l - K_{l-1})^{1-\varphi_l} (s - K_{l-1})^{\varphi_l-1} \frac{1}{\Gamma(\varphi_l)} \int_{K_{l-1}}^{K_l} (K_l - \alpha)^{\varphi_l-1} \xi(\alpha) d\alpha \\ &\quad - \frac{1}{\Gamma(\varphi_l)} \int_{K_{l-1}}^s (s - \alpha)^{\varphi_l-1} \xi(\alpha) d\alpha, \\ &= \frac{1}{\Gamma(\varphi_l)} \left[\int_{K_{l-1}}^s \left((K_l - K_{l-1})^{1-\varphi_l} (s - K_{l-1})^{\varphi_l-1} (K_l - \alpha)^{\varphi_l-1} - (s - \alpha)^{\varphi_l-1} \right) \xi(\alpha) d\alpha \right. \\ &\quad \left. + \int_s^{K_l} (K_l - K_{l-1})^{1-\varphi_l} (s - K_{l-1})^{\varphi_l-1} (K_l - \alpha)^{\varphi_l-1} \xi(\alpha) d\alpha \right]. \end{aligned}$$

The continuation of the function of the Green suggests that:

$$\xi(s) = \int_{K_{l-1}}^{K_l} G_l(s, \alpha) h \left(\alpha, \int_{K_{l-1}}^{K_l} G_l(s, \sigma) \xi(\sigma) d\sigma, \xi(\alpha) \right) d\alpha.$$

Let $\xi \in \Xi_l$ be the solution of the integral equation (7). Based on the continuity of the function $s^\delta h$ and the Lemma(2.1), we can conclude that ξ is the solution to the problem (6).

The next proposition will be necessary.

Proposition 3.1. ([30]) Assume that $s^\delta h : \Upsilon \times \Lambda \times \Lambda \rightarrow \Lambda$, ($0 \leq \delta \leq \min_{s \in \Upsilon} |(\varphi(s))|$) is a function that is continuous, $\varphi(s) : \Upsilon \rightarrow (1, 2]$ satisfies (H1). Then, the Green functions of the boundary value problem (6) satisfy the following characteristics:

- (1) $G_l(s, \alpha) \geq 0 \forall K_{l-1} \leq s, \quad \alpha \leq K_l.$
- (2) $\max_{s \in \Upsilon_l} G_l(s, \alpha) = G_l(\alpha, \alpha), \quad \alpha \in \Upsilon_l.$
- (3) $G_l(\alpha, \alpha)$ has one unique maximum given by:

$$\max_{\alpha \in \Upsilon_l} G_l(\alpha, \alpha) = \frac{1}{\Gamma(\varphi_l)} \left(\frac{K_l - K_{l-1}}{4} \right)^{\varphi_l-1}, \quad \text{for } l = 1, 2, \dots, n.$$

Theorem (2.1) is the basis for our first existence result.

Theorem 3.1. Let the conditions (H1)-(H2) to be satisfied, and if

$$\frac{(K_l - K_{l-1})^{\varphi_l-1} \left(K_l^{1-\delta} - K_{l-1}^{1-\delta} \right)}{(1 - \delta) 4^{\varphi_l-1} \Gamma(\varphi_l)} \left(\frac{\gamma_1 (K_l - K_{l-1})^{\varphi_l}}{4^{\varphi_l-1} \Gamma(\varphi_l)} + \gamma_2 \right) < 1, \tag{8}$$

the problem (6) has thus at least one solution on Υ .

Proof. We establish the operator

$$S : \Xi_\iota \rightarrow \Xi_\iota,$$

following:

$$S\xi(s) = \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \alpha)\xi(\alpha) d\alpha, \quad s \in \Upsilon_\iota, \tag{9}$$

where

$$\xi(\alpha) = h \left(\alpha, \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma)\xi(\sigma) d\sigma, \xi(\alpha) \right).$$

The operator S defined in (9) is well defined from the continuity of function $s^\delta h$ and the properties of fractional integrals.

Let,

$$R_\iota \geq \frac{\frac{h^*(K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota-1}\Gamma(\varphi_\iota)}}{1 - \frac{(K_\iota - K_{\iota-1})^{\varphi_\iota-1} (K_\iota^{1-\delta} - K_{\iota-1}^{1-\delta})}{(1-\delta)4^{\varphi_\iota-1}\Gamma(\varphi_\iota)} \left(\frac{\gamma_1(K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota-1}\Gamma(\varphi_\iota)} + \gamma_2 \right)},$$

with

$$h^* = \sup_{s \in \Upsilon_\iota} \|h(s, 0, 0)\|.$$

We evaluate the set

$$B_{R_\iota} = \left\{ \xi \in \Xi_\iota, \quad \|\xi\|_{\Xi_\iota} \leq R_\iota \right\},$$

where B_{R_ι} is bounded, convex, closed and nonempty. □

Now, we will demonstrate the satisfaction of Theorem (2.1) in four steps as outlined below.

STEP 1: $S(B_{R_\iota}) \subseteq (B_{R_\iota})$.

By Proposition (3.1) and (H2), we have

$$\begin{aligned} \|S\xi(s)\| &= \left\| \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \alpha)\xi(\alpha) d\alpha \right\|, \\ &\leq \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \alpha)\|\xi(\alpha)\| d\alpha, \\ &\leq \frac{1}{\Gamma(\varphi_\iota)} \left(\frac{K_\iota - K_{\iota-1}}{4} \right)^{\varphi_\iota-1} \int_{K_{\iota-1}}^{K_\iota} \|\xi(\alpha)\| d\alpha, \\ &\leq \frac{1}{\Gamma(\varphi_\iota)} \left(\frac{K_\iota - K_{\iota-1}}{4} \right)^{\varphi_\iota-1} \int_{K_{\iota-1}}^{K_\iota} \left\| h \left(\alpha, \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma)\xi(\sigma) d\sigma, \xi(\alpha) \right) \right\| d\alpha, \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{\Gamma(\varphi_\iota)} \left(\frac{K_\iota - K_{\iota-1}}{4} \right)^{\varphi_\iota-1} \int_{K_{\iota-1}}^{K_\iota} \left\| h \left(\alpha, \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) \xi(\sigma) d\sigma, y(\alpha) \right) - h(\alpha, 0, 0) \right\| d\alpha \\
 &\quad + \frac{1}{\Gamma(\varphi_\iota)} \left(\frac{K_\iota - K_{\iota-1}}{4} \right)^{\varphi_\iota-1} \int_{K_{\iota-1}}^{K_\iota} \|h(\alpha, 0, 0)\| d\alpha, \\
 &\leq \frac{1}{\Gamma(\varphi_\iota)} \left(\frac{K_\iota - K_{\iota-1}}{4} \right)^{\varphi_\iota-1} \int_{K_{\iota-1}}^{K_\iota} \alpha^{-\delta} \left(\gamma_1 \left\| \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) \xi(\sigma) d\sigma \right\| + \gamma_2 \|\xi(\alpha)\| \right) d\alpha \\
 &\quad + \frac{h^*(K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota-1} \Gamma(\varphi_\iota)}, \\
 &\leq \frac{1}{\Gamma(\varphi_\iota)} \left(\frac{K_\iota - K_{\iota-1}}{4} \right)^{\varphi_\iota-1} \int_{K_{\iota-1}}^{K_\iota} \alpha^{-\delta} \left(\frac{\gamma_1 (K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota-1} \Gamma(\varphi_\iota)} \|\xi\|_{\Xi_\iota} + \gamma_2 \|\xi\|_{\Xi_\iota} \right) d\alpha \\
 &\quad + \frac{h^*(K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota-1} \Gamma(\varphi_\iota)}, \\
 &\leq \frac{(K_\iota - K_{\iota-1})^{\varphi_\iota-1} (K_\iota^{1-\delta} - K_{\iota-1}^{1-\delta})}{(1 - \delta) 4^{\varphi_\iota-1} \Gamma(\varphi_\iota)} \left(\frac{\gamma_1 (K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota-1} \Gamma(\varphi_\iota)} + \gamma_2 \right) R_\iota + \frac{h^*(K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota-1} \Gamma(\varphi_\iota)}, \\
 &\leq R_\iota.
 \end{aligned}$$

It indicates $S(B_{R_\iota}) \subseteq B_{R_\iota}$.

STEP 2: S is continuous.

The sequence (ξ_n) is assumed to converge to ξ in Ξ_ι and $s \in \Upsilon_\iota$. Then,

$$\begin{aligned}
 \|(S\xi_n)(s) - (S\xi)(s)\| &\leq \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \alpha) \|\xi_n(\alpha) - \xi(\alpha)\| d\alpha, \\
 &\leq \frac{1}{\Gamma(\varphi_\iota)} \left(\frac{K_\iota - K_{\iota-1}}{4} \right)^{\varphi_\iota-1} \int_{K_{\iota-1}}^{K_\iota} \|\xi_n(\alpha) - \xi(\alpha)\| d\alpha, \\
 &\leq \frac{1}{\Gamma(\varphi_\iota)} \left(\frac{K_\iota - K_{\iota-1}}{4} \right)^{\varphi_\iota-1} \int_{K_{\iota-1}}^{K_\iota} \left\| h \left(\alpha, \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) \xi_n(\sigma) d\sigma, \xi_n(\alpha) \right) \right. \\
 &\quad \left. - h \left(\alpha, \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) y(\sigma) d\sigma, \xi(\alpha) \right) \right\| d\alpha, \\
 &\leq \frac{1}{\Gamma(\varphi_\iota)} \left(\frac{K_\iota - K_{\iota-1}}{4} \right)^{\varphi_\iota-1} \int_{K_{\iota-1}}^{K_\iota} \alpha^{-\delta} \left(\gamma_1 \left\| \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) \xi_n(\sigma) d\sigma \right. \right. \\
 &\quad \left. \left. - \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) \xi(\sigma) d\sigma \right\| + \gamma_2 \|\xi_n(\alpha) - \xi(\alpha)\| \right) d\alpha, \\
 &\leq \frac{1}{\Gamma(\varphi_\iota)} \left(\frac{K_\iota - K_{\iota-1}}{4} \right)^{\varphi_\iota-1} \int_{K_{\iota-1}}^{K_\iota} \alpha^{-\delta} \left(\gamma_1 \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) \|\xi_n(\sigma) - \xi(\sigma)\| d\sigma \right. \\
 &\quad \left. + \gamma_2 \|\xi_n(\alpha) - \xi(\alpha)\| \right) d\alpha,
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(\varphi_\iota)} \left(\frac{K_\iota - K_{\iota-1}}{4} \right)^{\varphi_\iota - 1} \int_{K_{\iota-1}}^{K_\iota} \alpha^{-\delta} \left(\frac{\gamma_1(K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota - 1} \Gamma(\varphi_\iota)} \|\xi_n - \xi\|_{\Xi_\iota} \right. \\ &\quad \left. + \gamma_2 \|\xi_n - \xi\|_{\Xi_\iota} \right) d\alpha, \\ &\leq \frac{(K_\iota - K_{\iota-1})^{\varphi_\iota - 1} (K_\iota^{1-\delta} - K_{\iota-1}^{1-\delta})}{(1 - \delta) 4^{\varphi_\iota - 1} \Gamma(\varphi_\iota)} \left(\frac{\gamma_1(K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota - 1} \Gamma(\varphi_\iota)} + \gamma_2 \right) \|\xi_n - \xi\|_{\Xi_\iota}, \end{aligned}$$

i.e., we get

$$\|(S\xi_n) - (S\xi)\|_{\Xi_\iota} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, S is a continuous operator.

STEP 3: S is bounded and equicontinuous.

Step 2 yields $S(B_{R_\iota}) = \{S(\xi) : \xi \in B_{R_\iota}\} \subset B_{R_\iota}$. Consequently, for each $\xi \in B_{R_\iota}$, $\|S(\xi)\|_{\Xi_\iota} \leq R_\iota$ holds, showing that $S(B_{R_\iota})$ is bounded. It remains essential to establish that $S(B_{R_\iota})$ is equicontinuous.

For $s_1, s_2 \in \Upsilon_\iota$, $s_1 < s_2$ and $\xi \in B_{R_\iota}$, we obtain

$$\begin{aligned} \|(S\xi)(s_2) - (S\xi)(s_1)\| &= \left\| \int_{K_{\iota-1}}^{K_\iota} G_\iota(s_2, \alpha) \xi(\alpha) d\alpha - \int_{K_{\iota-1}}^{K_\iota} G_\iota(s_1, \alpha) \xi(\alpha) d\alpha \right\|, \\ &\leq \int_{K_{\iota-1}}^{K_\iota} \left\| (G_\iota(s_2, \alpha) - G_\iota(s_1, \alpha)) \xi(\alpha) \right\| d\alpha, \\ &\leq \int_{K_{\iota-1}}^{K_\iota} \|G_\iota(s_2, \alpha) - G_\iota(s_1, \alpha)\| \left\| h \left(\alpha, \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) \xi(\sigma) d\sigma, \xi(\alpha) \right) \right\| d\alpha, \\ &\leq \int_{K_{\iota-1}}^{K_\iota} \|G_\iota(s_2, \alpha) - G_\iota(s_1, \alpha)\| \left(\left\| h \left(\alpha, \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) \xi(\sigma) d\sigma, \xi(\alpha) \right) \right. \right. \\ &\quad \left. \left. - h(\alpha, 0, 0) \right\| + \|h(\alpha, 0, 0)\| \right) d\alpha, \\ &\leq \int_{K_{\iota-1}}^{K_\iota} \|G_\iota(s_2, \alpha) - G_\iota(s_1, \alpha)\| \left[\alpha^{-\delta} \left(\gamma_1 \left\| \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) \xi(\sigma) d\sigma \right\| \right. \right. \\ &\quad \left. \left. + \gamma_2 \|\xi(\alpha)\| \right) + h^* \right] d\alpha, \\ &\leq \int_{K_{\iota-1}}^{K_\iota} \|G_\iota(s_2, \alpha) - G_\iota(s_1, \alpha)\| \left[\alpha^{-\delta} \left(\frac{\gamma_1(K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota - 1} \Gamma(\varphi_\iota)} \|\xi\|_{\Xi_\iota} + \gamma_2 \|\xi\|_{\Xi_\iota} \right) \right. \\ &\quad \left. + h^* \right] d\alpha, \\ &\leq \left(\frac{\gamma_1(K_\iota - K_{\iota-1})^{\varphi_\iota} (K_{\iota-1})^{-\delta}}{4^{\varphi_\iota - 1} \Gamma(\varphi_\iota)} + \gamma_2 (K_{\iota-1})^{-\delta} \right) \|\xi\|_{\Xi_\iota} \int_{K_{\iota-1}}^{K_\iota} \|G_\iota(s_2, \alpha) \\ &\quad - G_\iota(s_1, \alpha)\| d\alpha + h^* \int_{K_{\iota-1}}^{K_\iota} \|G_\iota(s_2, \alpha) - G_\iota(s_1, \alpha)\| d\alpha, \end{aligned}$$

using the continuity of Green’s function. Hence $\|(S\xi)(s_2) - (S\xi)(s_1)\|_{E_\iota} \rightarrow 0$ as $|s_2 - s_1| \rightarrow 0$. It implies that $S(B_{R_\iota})$ is equicontinuous.

STEP 4: S is ι -set contractions.

If $\chi \in B_{R_\iota}, s \in \Upsilon_\iota$, we receive,

$$\vartheta(S(\chi)(s)) = \vartheta((S\xi)(s), \xi \in \chi),$$

$$\leq \left\{ \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \alpha) \vartheta h \left(\alpha, \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) \xi(\sigma) d\sigma, \xi(\alpha) \right) d\alpha, \quad \xi \in \chi \right\}.$$

Then, according to Remark 3.1, we have

$$\vartheta(S(\chi)(s)) \leq \left\{ \int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \alpha) \alpha^{-\delta} \left[\gamma_1 \vartheta \left(\int_{K_{\iota-1}}^{K_\iota} G_\iota(s, \sigma) \xi(\sigma) d\sigma \right) + \gamma_2 \vartheta(\xi(\alpha)) \right] d\alpha, \quad y \in \chi \right\},$$

$$\leq \left\{ \frac{1}{\Gamma(\varphi_\iota)} \left(\frac{K_\iota - K_{\iota-1}}{4} \right)^{\varphi_\iota - 1} \left[\frac{\gamma_1 (K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota - 1} \Gamma(\varphi_\iota)} \widehat{\vartheta}(\chi) \int_{K_{\iota-1}}^{K_\iota} \alpha^{-\delta} d\alpha \right. \right.$$

$$\left. \left. + \gamma_2 \widehat{\vartheta}(\chi) \int_{K_{\iota-1}}^{K_\iota} \alpha^{-\delta} d\alpha \right], \quad \xi \in \chi \right\},$$

$$\leq \frac{(K_\iota^{1-\delta} - K_{\iota-1}^{1-\delta})(K_\iota - K_{\iota-1})^{\varphi_\iota - 1}}{4^{\varphi_\iota - 1} (1 - \delta) \Gamma(\varphi_\iota)} \left(\frac{\gamma_1 (K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota - 1} \Gamma(\varphi_\iota)} + \gamma_2 \right) \widehat{\vartheta}(\chi).$$

Therefore,

$$\widehat{\vartheta}(S\chi) \leq \frac{(K_\iota^{1-\delta} - K_{\iota-1}^{1-\delta})(K_\iota - K_{\iota-1})^{\varphi_\iota - 1}}{4^{\varphi_\iota - 1} (1 - \delta) \Gamma(\varphi_\iota)} \left(\frac{\gamma_1 (K_\iota - K_{\iota-1})^{\varphi_\iota}}{4^{\varphi_\iota - 1} \Gamma(\varphi_\iota)} + \gamma_2 \right) \widehat{\vartheta}(\chi).$$

As a result of (8), we conclude that S is a ι -set contraction. According to Theorem (2.1), problem (6) has at least one solution \tilde{y}_ι in B_{R_ι} .

Let

$$\xi_\iota = \begin{cases} 0, & s \in [0, K_{\iota-1}], \\ \tilde{\xi}_\iota, & s \in \Upsilon_\iota. \end{cases} \tag{10}$$

$\xi_\iota \in C([0, K_\iota], \Lambda)$ is defined by (10) and known to satisfy equation:

$$\frac{d^2}{ds^2} \left(\int_0^{K_1} \frac{(s - \alpha)^{1-\varphi_1}}{\Gamma(2 - \varphi_1)} \xi_\iota(\alpha) d\alpha + \dots + \int_{K_{\iota-1}}^s \frac{(s - \alpha)^{1-\varphi_\iota}}{\Gamma(2 - \varphi_\iota)} \xi_\iota(\alpha) d\alpha \right) + h(\alpha, \xi_\iota(\alpha), D_{0+}^{\varphi_\iota} \xi_\iota(\alpha)) = 0,$$

for $s \in \Upsilon_\iota$, implying that ξ_ι is a solution of (5) with $\xi_\iota(0) = 0, \xi_\iota(K_\iota) = \tilde{\xi}_\iota(K_\iota) = 0$.

Then,

$$\xi(s) = \begin{cases} \xi_1(s), & s \in \Upsilon_1, \\ \xi_2(s) = \begin{cases} 0, & s \in \Upsilon_1, \\ \tilde{\xi}_2, & s \in \Upsilon_2, \end{cases} \\ \vdots \\ \xi_n(s) = \begin{cases} 0, & s \in [0, K_{l-1}], \\ \tilde{\xi}_l, & s \in \Upsilon_l. \end{cases} \end{cases}$$

It constitutes a solution of BVP(1).

4 Hyers-Ulam-Rassias Stability

Theorem 4.1. Consider (H1), (H2), (8) and

(H3) Let $\psi \in C(\Upsilon_l, \Lambda)$ is an increasing function, and $\lambda_\psi > 0$ exists such that

$$I_{K_{l-1}^+}^{\varphi_l} \psi(s) \leq \lambda_\psi \psi(s), \text{ for any } s \in \Upsilon_l.$$

The equation of (1) is hence (HUR) stable with respect to ψ .

Proof. Consider $z \in C(\Upsilon_l, \Lambda)$ to be a solution to the following inequality:

$$\left\| D_{K_{l-1}^+}^{\varphi_l} z(s) + h\left(s, z(s), D_{K_{l-1}^+}^{\varphi_l} z(s)\right) \right\| \leq \epsilon \psi(s), \quad s \in \Upsilon_l. \tag{11}$$

$\xi \in C(\Upsilon_l, \Lambda)$ is a solution to the given problem:

$$\begin{cases} D_{K_{l-1}^+}^{\varphi_l} \xi(s) + h\left(s, \xi(s), D_{K_{l-1}^+}^{\varphi_l} \xi(s)\right) = 0, & s \in \Upsilon_l, \\ \xi(K_{l-1}) = 0, & \xi(K_l) = 0. \end{cases}$$

By using Lemma (3.1), we have

$$\xi(s) = \int_{K_{l-1}}^{K_l} G_l(s, \alpha) h\left(\alpha, \int_{K_{l-1}}^{K_l} G_l(s, \sigma) \xi(\sigma) d\sigma, \xi(\alpha)\right) d\alpha.$$

From the integration of (11) and (H3), we get

$$\begin{aligned} \left\| z(s) + \int_{K_{l-1}}^{K_l} G_l(s, \alpha) h\left(\alpha, \int_{K_{l-1}}^{K_l} G_l(s, \sigma) z(\sigma) d\sigma, z(\alpha)\right) d\alpha \right\| &\leq \epsilon \int_{K_{l-1}}^s \frac{(s-\alpha)^{\varphi(l)-1}}{\Gamma(\varphi(l))} \psi(\alpha) d\alpha, \\ &\leq \epsilon \lambda_\psi \psi(s). \end{aligned}$$

On the other hand, by Proposition (3.1), we have, for each $s \in \Upsilon_l$

$$\begin{aligned} \|z(s) - \xi(s)\| &\leq \left\| z(s) - \int_{K_{l-1}}^{K_l} G_l(s, \alpha) h \left(\alpha, \int_{K_{l-1}}^{K_l} G_l(s, \sigma) z(\sigma) d\sigma, z(\alpha) \right) d\alpha \right\| \\ &\quad + \int_{K_{l-1}}^{K_l} G_l(s, \alpha) \left\| h \left(\alpha, \int_{K_{l-1}}^{K_l} G_l(s, \sigma) z(\sigma) d\sigma, z(\alpha) \right) \right. \\ &\quad \left. - h \left(\alpha, \int_{K_{l-1}}^{K_l} G_l(s, \sigma) y(\sigma) d\sigma, \xi(\alpha) \right) \right\| d\alpha, \\ &\leq \left\| z(s) + \int_{K_{l-1}}^{K_l} G_l(s, \alpha) h \left(\alpha, \int_{K_{l-1}}^{K_l} G_l(s, \sigma) z(\sigma) d\sigma, z(\alpha) \right) d\alpha \right\| \\ &\quad + \int_{K_{l-1}}^{K_l} G_l(s, \alpha) \left\| h \left(\alpha, \int_{K_{l-1}}^{K_l} G_l(s, \sigma) z(\sigma) d\sigma, z(\alpha) \right) \right. \\ &\quad \left. - h \left(\alpha, \int_{K_{l-1}}^{K_l} G_l(s, \sigma) y(\sigma) d\sigma, \xi(\alpha) \right) \right\| d\alpha, \\ &\leq \lambda_{\psi(s)} \epsilon \psi(s) \\ &\quad + \frac{1}{\Gamma(\varphi_l)} \left(\frac{K_l - K_{l-1}}{4} \right)^{\varphi_l - 1} \int_{K_{l-1}}^{K_l} \alpha^{-\delta} \left(\gamma_1 \int_{K_{l-1}}^{K_l} G_l(s, \sigma) \|z(\sigma) - \xi(\sigma)\| d\sigma \right. \\ &\quad \left. + \gamma_2 \|z(\alpha) - \xi(\alpha)\| \right) d\alpha, \\ &\leq \lambda_{\psi(s)} \epsilon \psi(s) + \frac{1}{\Gamma(\varphi_l)} \left(\frac{K_l - K_{l-1}}{4} \right)^{\varphi_l - 1} \int_{K_{l-1}}^{K_l} \alpha^{-\delta} \left(\gamma_1 \frac{(K_l - K_{l-1})^{\varphi_l}}{4^{\varphi_l - 1} \Gamma(\varphi_l)} \|z - \xi\|_{\Xi_l} \right. \\ &\quad \left. + \gamma_2 \|z - \xi\|_{\Xi_l} \right) d\alpha, \\ &\leq \lambda_{\psi(s)} \epsilon \psi(s) + \frac{(K_l - K_{l-1})^{\varphi_l - 1} (K_l^{1-\delta} - K_{l-1}^{1-\delta})}{(1 - \delta) 4^{\varphi_l - 1} \Gamma(\varphi_l)} \left(\frac{\gamma_1 (K_l - K_{l-1})^{\varphi_l}}{4^{\varphi_l - 1} \Gamma(\varphi_l)} + \gamma_2 \right) \|z - \xi\|_{\Xi_l}. \end{aligned}$$

Hence,

$$\|z - \xi\|_{E_l} \left[1 - \frac{(K_l - K_{l-1})^{\varphi_l - 1} (K_l^{1-\delta} - K_{l-1}^{1-\delta})}{(1 - \delta) 4^{\varphi_l - 1} \Gamma(\varphi_l)} \left(\frac{\gamma_1 (K_l - K_{l-1})^{\varphi_l}}{4^{\varphi_l - 1} \Gamma(\varphi_l)} + \gamma_2 \right) \right] \leq \lambda_{\psi(s)} \epsilon \psi(s).$$

We get, for any $s \in \Upsilon_l$

$$\begin{aligned} \|z - \xi\|_{\Xi_l} &\leq \left[1 - \frac{(K_l - K_{l-1})^{\varphi_l - 1} (K_l^{1-\delta} - K_{l-1}^{1-\delta})}{(1 - \delta) 4^{\varphi_l - 1} \Gamma(\varphi_l)} \left(\frac{\gamma_1 (K_l - K_{l-1})^{\varphi_l}}{4^{\varphi_l - 1} \Gamma(\varphi_l)} + \gamma_2 \right) \right]^{-1} \lambda_{\psi(s)} \epsilon \psi(s), \\ &:= c_g \epsilon \psi(s). \end{aligned}$$

Then, BVP (1) is (HUR) stable. □

5 Approximate Numerical Examples

5.1 Example 5.1

Consider the following BVP:

$$\begin{cases} D_{0+}^{\varphi(s)} \xi(s) + \frac{|\xi(s)| + |D_{0+}^{\varphi(s)} \xi(s)|}{(s+1)^{\frac{1}{2}}(9+e^s)(1+\xi^2(s))} = 0, & s \in \Upsilon := [0, 2], \\ \xi(0) = 0, & \xi(2) = 0. \end{cases} \tag{12}$$

Let,

$$h(s, \pi, z) = \frac{\pi + z}{(s+1)^{\frac{1}{2}}(9+e^s)(1+\pi^2)}, \quad (s, \pi, z) \in [0, 2] \times [0, +\infty) \times [0, +\infty).$$

$$\varphi(s) = \begin{cases} \frac{9}{8}, & s \in \Upsilon_1 := [0, 1], \\ \frac{7}{4}, & s \in \Upsilon_2 :=]1, 2]. \end{cases} \tag{13}$$

Then, we have

$$\begin{aligned} s^{\frac{1}{2}} \left| h(s, \pi_1, \varsigma_1) - h(s, \pi_2, \varsigma_2) \right| &\leq \frac{1}{9+e^t} \left| \frac{\pi_1 + \varsigma_1}{1 + \pi_1^2} - \frac{\pi_2 + \varsigma_2}{1 + \pi_2^2} \right|, \\ &\leq \frac{1}{10} \left| (\pi_1 + \varsigma_1) - (\pi_2 + \varsigma_2) \right|, \\ &\leq \frac{1}{10} |\pi_1 - \pi_2| + \frac{1}{10} |\varsigma_1 - \varsigma_2|. \end{aligned}$$

Consequently, (H2) holds if $\delta = \frac{1}{2}, \gamma_1 = \gamma_2 = \frac{1}{10}$.

The problem’s equation (12) is decomposed into two expressions by (13).

$$\begin{cases} D_{0+}^{\frac{9}{8}} \xi(s) + \frac{|\xi(s)| + |D_{0+}^{\frac{9}{8}} \xi(s)|}{(s+1)^{\frac{1}{2}}(9+e^s)(1+\xi^2(s))} = 0, & s \in \Upsilon_1, \\ D_{1+}^{\frac{7}{4}} \xi(s) + \frac{|\xi(s)| + |D_{1+}^{\frac{7}{4}} \xi(s)|}{(s+1)^{\frac{1}{2}}(9+e^s)(1+\xi^2(s))} = 0, & s \in \Upsilon_2. \end{cases}$$

For $s \in \Upsilon_1$, (12) corresponds to the next problem:

$$\begin{cases} D_{0+}^{\frac{9}{8}} \xi(s) + \frac{|\xi(s)| + |D_{0+}^{\frac{9}{8}} \xi(s)|}{(s+1)^{\frac{1}{2}}(9+e^s)(1+\xi^2(s))} = 0, & s \in \Upsilon_1, \\ \xi(0) = 0, & \xi(1) = 0. \end{cases} \tag{14}$$

Next, we demonstrate that the condition (8) is satisfied

$$\begin{aligned} \frac{(K_1 - K_0)^{\varphi_1 - 1} (K_1^{1-\delta} - K_0^{1-\delta})}{(1 - \delta)4^{\varphi_1 - 1} \Gamma(\varphi_1)} \left(\frac{\gamma_1 (K_1 - K_0)^{\varphi_1}}{4^{\varphi_1 - 1} \Gamma(\varphi_1)} + \gamma_2 \right) &= \frac{(1)^{\frac{1}{8}} (1^{\frac{1}{2}})}{\frac{1}{2} 4^{\frac{1}{8}} \Gamma(\frac{9}{8})} \left(\frac{\frac{1}{10} (1)^{\frac{9}{8}}}{4^{\frac{1}{8}} \Gamma(\frac{9}{8})} + \frac{1}{10} \right), \\ &\simeq 0.3380 < 1. \end{aligned}$$

Let $\psi(s) = s^{\frac{1}{2}}$,

$$\begin{aligned} I_{0+}^{\varphi_1} \psi(t) &= \frac{1}{\Gamma(\frac{9}{8})} \int_0^s (s - \alpha)^{\frac{1}{8}} \alpha^{\frac{1}{2}} d\alpha, \\ &\leq \frac{1}{\Gamma(\frac{9}{8})} \int_0^s (s - \alpha)^{\frac{1}{8}} d\alpha, \\ &\leq \frac{8}{9\Gamma(\frac{9}{8})} \psi(s). \end{aligned}$$

Consequently, if $\psi(s) = s^{\frac{1}{2}}$ and $\lambda_{\psi(s)} = \frac{8}{9\Gamma(\frac{9}{8})}$, then (H3) is satisfied.

The problem (14) has a solution $\xi_1 \in \Xi_1$ according to the Theorem (3.1).

The problem (12) may be formulated as follows for $s \in \Upsilon_2$:

$$\begin{cases} D_{1+}^{\frac{7}{4}} \xi(s) + \frac{|\xi(s)| + |D_{1+}^{\frac{7}{4}} \xi(s)|}{(s+1)^{\frac{1}{2}}(9+e^s)(1+\xi^2(s))} = 0, & s \in \Upsilon_2, \\ \xi(1) = 0, \quad \xi(2) = 0. \end{cases} \tag{15}$$

We see that,

$$\begin{aligned} \frac{(K_2 - K_1)^{\varphi_2 - 1} (K_2^{1-\delta} - K_1^{1-\delta})}{(1 - \delta)4^{\varphi_2 - 1}\Gamma(\varphi_2)} \left(\frac{\gamma_1 (K_2 - K_1)^{\varphi_2}}{4^{\varphi_2 - 1}\Gamma(\varphi_2)} + \gamma_2 \right) &= \frac{(1)^{\frac{3}{4}}(2^{\frac{1}{2}} - 1^{\frac{1}{2}})}{\frac{1}{2}4^{\frac{3}{4}}\Gamma(\frac{7}{4})} \left(\frac{\frac{1}{10}(1)^{\frac{7}{4}}}{4^{\frac{3}{4}}\Gamma(\frac{7}{4})} + \frac{1}{10} \right), \\ &\simeq 0.0441 < 1. \end{aligned}$$

Therefore, (8) is satisfied, and

$$\begin{aligned} I_{1+}^{\varphi_2} \psi(s) &= \frac{1}{\Gamma(\frac{7}{4})} \int_1^s (s - \alpha)^{\frac{3}{4}} \alpha^{\frac{1}{2}} d\alpha, \\ &\leq \frac{\sqrt{2}}{\Gamma(\frac{7}{4})} \int_1^s (s - \alpha)^{\frac{3}{4}} d\alpha, \\ &\leq \frac{4\sqrt{2}}{7\Gamma(\frac{7}{4})} \psi(s) := \lambda_{\psi(s)} \psi(s). \end{aligned}$$

Consequently, (H3) is satisfied by $\psi(s) = s^{\frac{1}{2}}$ and $\lambda_{\psi(s)} = \frac{4\sqrt{2}}{7\Gamma(\frac{7}{4})}$.

The problem (15) has to have a solution $\tilde{\xi}_2 \in \Xi_2$ according to Theorem (3.1).

It's well known,

$$\xi_2(s) = \begin{cases} 0, & s \in \Upsilon_1, \\ \tilde{\xi}_2(s), & s \in \Upsilon_2. \end{cases}$$

As a result, by Definition (3.1), the boundary value problem (12) has a solution:

$$\xi(s) = \begin{cases} \xi_1(s), & s \in \Upsilon_1, \\ \xi_2(s) = \begin{cases} 0, & s \in \Upsilon_1, \\ \tilde{\xi}_2(s), & s \in \Upsilon_2. \end{cases} \end{cases}$$

Furthermore, according to Theorem (4.1), the equation of (12) is (HUR) stable.

In those figures, we present the plot of the function $h(s, \pi, z)$ in the two interval $[0, 1]$ and $[1, 2]$.

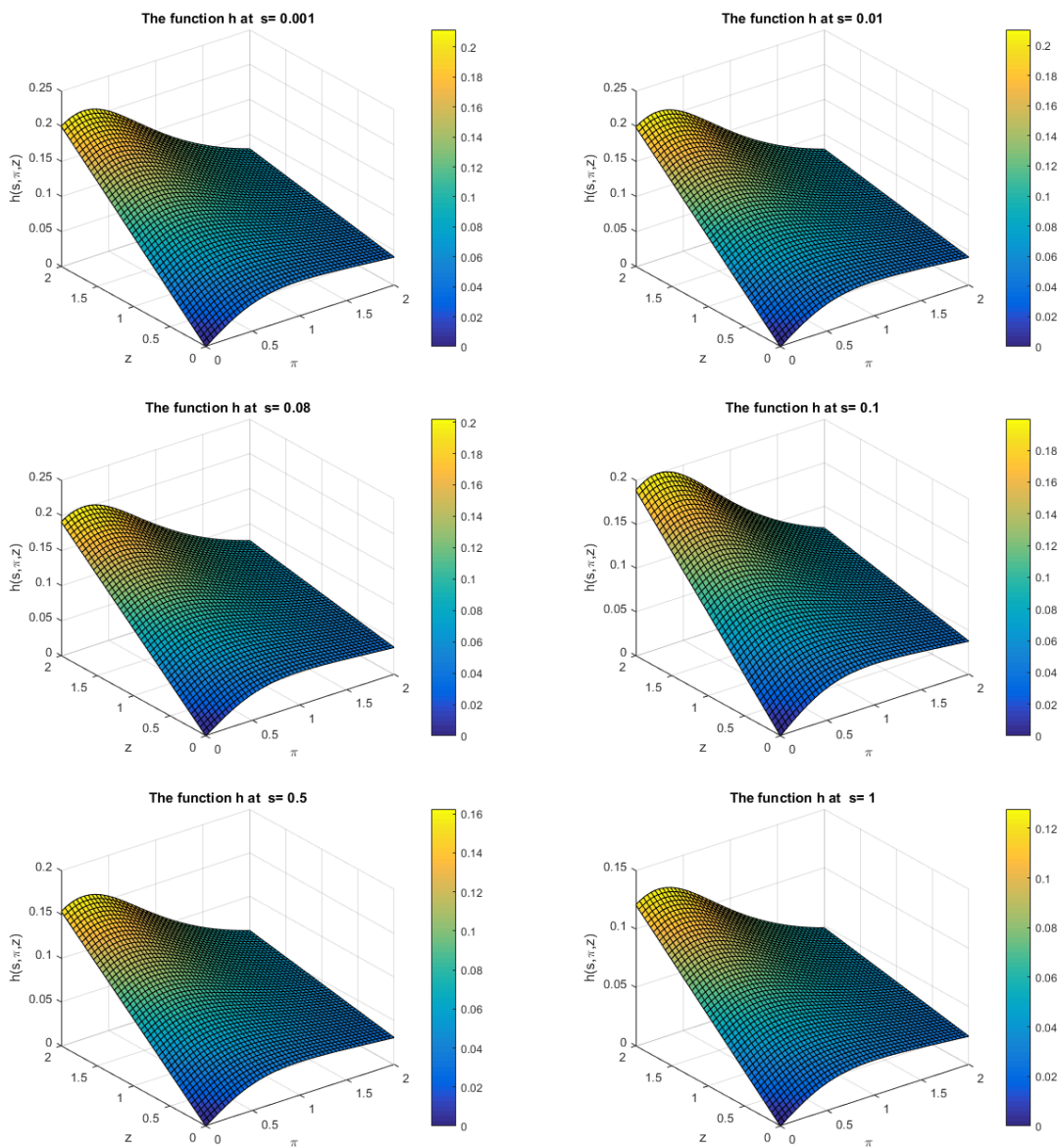


Figure 1: A plot of $h(s, \pi, z)$ for different values of $s \in [0, 1]$.

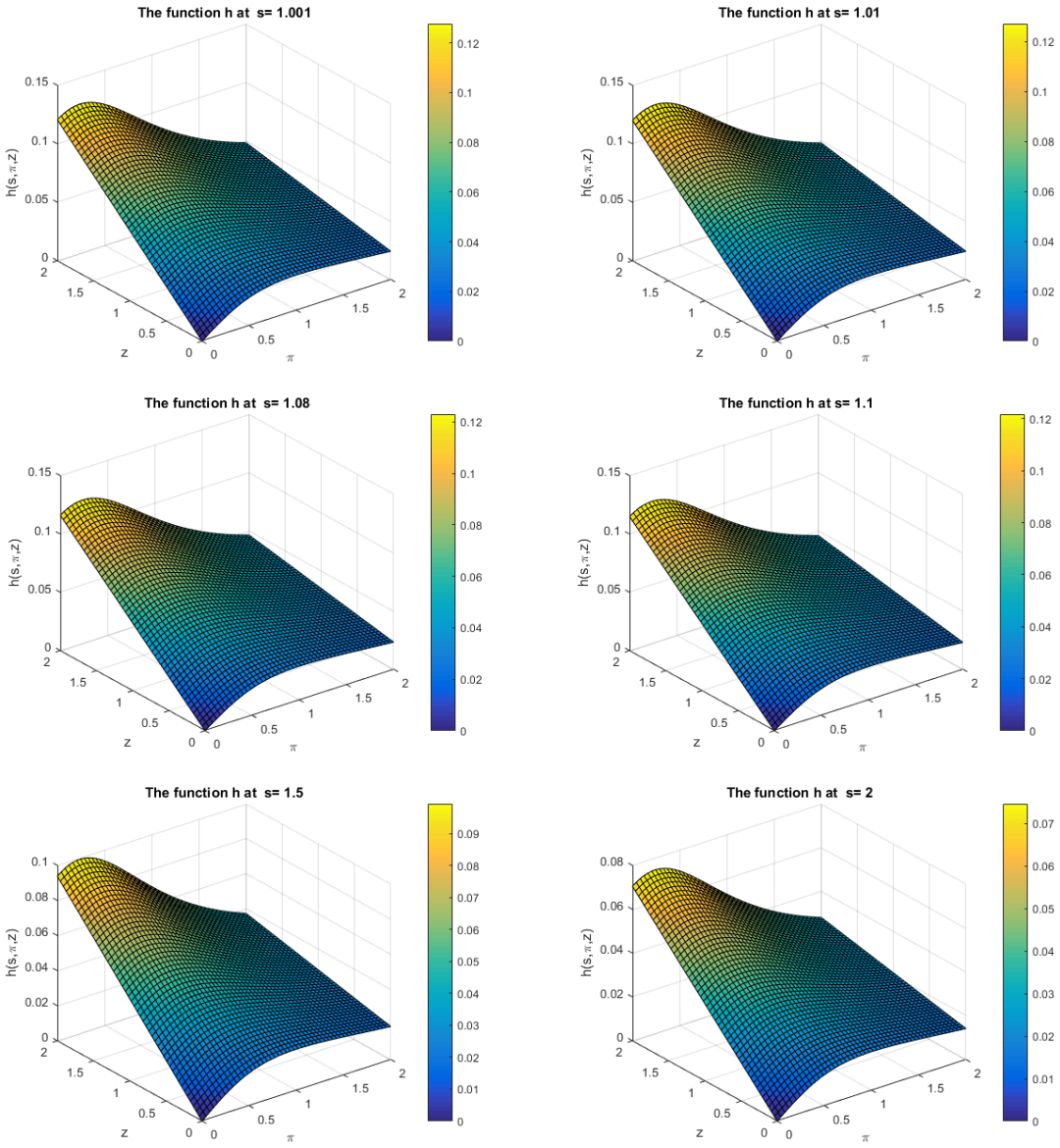


Figure 2: A plot of $h(s, \pi, z)$ for different values of $s \in]1, 2]$.

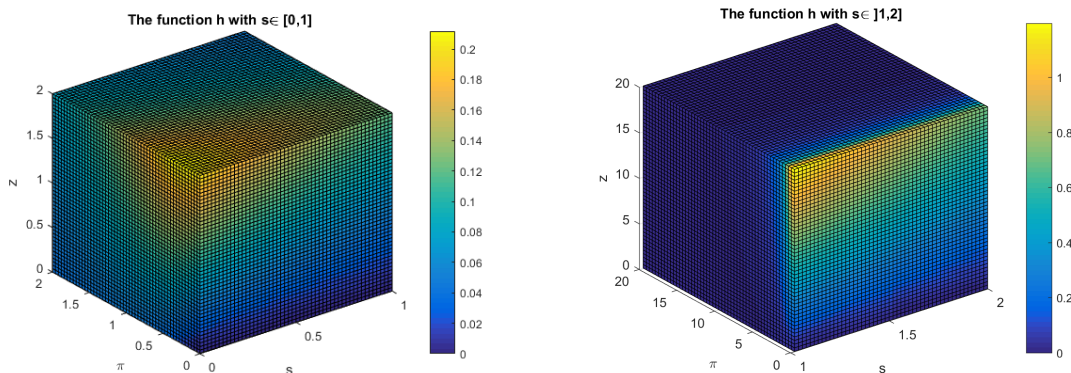


Figure 3: The 3D plot of $h(s, \pi, z)$ for $s \in [0, 1]$ and $]1, 2]$.

In this table, we present the value of $\max_{\pi, z} h(s, \pi, z)$ for different values of $s \in [0, 1]$ and $]1, 2]$.

$s \in [0, 1]$	0.0010	0.0100	0.0200	0.0800	0.1000	0.5000	1.0000
$\max_{\pi, z} h(s, \pi, z)$	0.2117	0.2105	0.2093	0.2021	0.1998	0.1624	0.1278
$s \in]1, 2]$	1.0010	1.0100	1.0200	1.0800	1.1000	1.5000	2.0000
$\max_{\pi, z} h(s, \pi, z)$	0.1277	0.1272	0.1266	0.1229	0.1217	0.0994	0.0746

We observe that the value of $\max_{\pi, z} h(s, \pi, z)$ is decreasing with respect to the value of s in the two intervals.

5.2 Example 5.2

Consider the following BVP:

$$\begin{cases} D_{0+}^{\varphi(s)} \xi(s) + \frac{s^{-\frac{1}{4}} e^{-s}}{(4e^{\frac{s}{1+s^2}} + 5e^s + 2)(1 + |\xi(s)| + |D_{0+}^{\varphi(s)} \xi(s)|)} = 0, & s \in \Upsilon := [0, 3], \\ \xi(0) = 0, \quad \xi(3) = 0. \end{cases} \tag{16}$$

Let,

$$h(s, \pi, z) = \frac{s^{-\frac{1}{4}} e^{-s}}{(4e^{\frac{s}{1+s^2}} + 5e^s + 2)(1 + \pi + z)}, \quad (s, \pi, z) \in [0, 3] \times [0, +\infty) \times [0, +\infty).$$

$$\varphi(s) = \begin{cases} \frac{3}{2}, & s \in \Upsilon_1 := [0, 1], \\ \frac{8}{5}, & s \in \Upsilon_2 :=]1, 2], \\ \frac{9}{5}, & s \in \Upsilon_3 :=]2, 3]. \end{cases} \tag{17}$$

Then, we have

$$\begin{aligned} s^{\frac{1}{4}} \left| h(s, \pi_1, \varsigma_1) - h(s, \pi_2, \varsigma_2) \right| &\leq \frac{1}{e^s (4e^{\frac{s}{1+s^2}} + 5e^s + 2)} \left| \frac{1}{1 + \pi_1 + \varsigma_1} - \frac{1}{1 + \pi_2 + \varsigma_2} \right|, \\ &\leq \frac{1}{11} \left| (\pi_2 + \varsigma_2) - (\pi_1 + \varsigma_1) \right|, \\ &\leq \frac{1}{11} |\pi_1 - \pi_2| + \frac{1}{11} |\varsigma_1 - \varsigma_2|. \end{aligned}$$

Consequently, (H2) holds if $\delta = \frac{1}{4}$, $\gamma_1 = \gamma_2 = \frac{1}{11}$.

The problem’s equation (16) is decomposed into two expressions by (17),

$$\begin{cases} D_{0+}^{\frac{3}{2}} \xi(s) + \frac{s^{-\frac{1}{4}} e^{-s}}{(4e^{\frac{s}{1+s^2}} + 5e^s + 2)(1 + |\xi(s)| + |D_{0+}^{\varphi(s)} \xi(s)|)} = 0, & s \in \Upsilon_1, \\ D_{1+}^{\frac{8}{5}} \xi(s) + \frac{s^{-\frac{1}{4}} e^{-s}}{(4e^{\frac{s}{1+s^2}} + 5e^s + 2)(1 + |\xi(s)| + |D_{1+}^{\varphi(s)} \xi(s)|)} = 0, & s \in \Upsilon_2, \\ D_{2+}^{\frac{9}{5}} \xi(s) + \frac{s^{-\frac{1}{4}} e^{-s}}{(4e^{\frac{s}{1+s^2}} + 5e^s + 2)(1 + |\xi(s)| + |D_{2+}^{\varphi(s)} \xi(s)|)} = 0, & s \in \Upsilon_3. \end{cases}$$

For $s \in \Upsilon_1$, (16) corresponds to the next problem:

$$\begin{cases} D_{0+}^{\frac{3}{2}} \xi(s) + \frac{s^{-\frac{1}{4}} e^{-s}}{(4e^{\frac{s}{1+s^2}} + 5e^s + 2)(1 + |\xi(s)| + |D_{0+}^{\varphi(s)} \xi(s)|)} = 0, & s \in \Upsilon_1, \\ \xi(0) = 0, \quad \xi(1) = 0. \end{cases} \tag{18}$$

Next, we demonstrate that the condition (8) is satisfied.

$$\begin{aligned} \frac{(K_1 - K_0)^{\varphi_1 - 1} (K_1^{1-\delta} - K_0^{1-\delta})}{(1 - \delta) 4^{\varphi_1 - 1} \Gamma(\varphi_1)} \left(\frac{\gamma_1 (K_1 - K_0)^{\varphi_1}}{4^{\varphi_1 - 1} \Gamma(\varphi_1)} + \gamma_2 \right) &= \frac{(1)^{\frac{1}{2}} (1^{\frac{3}{4}})}{\frac{3}{4} 4^{\frac{1}{2}} \Gamma(\frac{3}{2})} \left(\frac{\frac{1}{11} (1)^{\frac{3}{2}}}{4^{\frac{1}{2}} \Gamma(\frac{3}{2})} + \frac{1}{11} \right), \\ &\approx 0.1070 < 1. \end{aligned}$$

The problem (18) has a solution $\xi_1 \in \Xi_1$ according to the Theorem (3.1).

The problem (16) may be formulated as follows for $s \in \Upsilon_2$:

$$\begin{cases} D_{1+}^{\frac{8}{5}} \xi(s) + \frac{s^{-\frac{1}{4}} e^{-s}}{(4e^{\frac{s}{1+s^2}} + 5e^s + 2)(1 + |\xi(s)| + |D_{1+}^{\varphi(s)} \xi(s)|)} = 0, & s \in \Upsilon_2, \\ \xi(1) = 0, \quad \xi(2) = 0. \end{cases} \tag{19}$$

We see that

$$\frac{(K_2 - K_1)^{\varphi_2 - 1} (K_2^{1-\delta} - K_1^{1-\delta})}{(1 - \delta)4^{\varphi_2 - 1}\Gamma(\varphi_2)} \left(\frac{\gamma_1 (K_2 - K_1)^{\varphi_2}}{4^{\varphi_2 - 1}\Gamma(\varphi_2)} + \gamma_2 \right) = \frac{(1)^{\frac{3}{5}}(2^{\frac{3}{4}} - 1^{\frac{3}{4}})}{\frac{3}{4}4^{\frac{3}{5}}\Gamma(\frac{8}{5})} \left(\frac{1}{11}(1)^{\frac{8}{5}} + \frac{1}{11} \right),$$

$$\simeq 0.0599 < 1.$$

Therefore, (8) is satisfied.

The problem (20) has to have a solution $\tilde{\xi}_2 \in \Xi_2$ according to Theorem (3.1).

The problem (16) may be formulated as follows for $s \in \Upsilon_3$:

$$\begin{cases} D_{2^+}^{\frac{9}{5}}\xi(s) + \frac{s^{-\frac{1}{4}}e^{-s}}{(4e^{\frac{s}{1+s^2}} + 5e^s + 2)(1 + |\xi(s)| + |D_{2^+}^{\varphi(s)}\xi(s)|)} = 0, & s \in \Upsilon_3, \\ \xi(2) = 0, \quad \xi(3) = 0. \end{cases} \tag{20}$$

We see that

$$\frac{(K_3 - K_2)^{\varphi_3 - 1} (K_3^{1-\delta} - K_2^{1-\delta})}{(1 - \delta)4^{\varphi_3 - 1}\Gamma(\varphi_3)} \left(\frac{\gamma_1 (K_3 - K_2)^{\varphi_3}}{4^{\varphi_3 - 1}\Gamma(\varphi_3)} + \gamma_2 \right) = \frac{(1)^{\frac{4}{5}}(3^{\frac{3}{4}} - 2^{\frac{3}{4}})}{\frac{3}{4}4^{\frac{4}{5}}\Gamma(\frac{9}{5})} \left(\frac{1}{11}(1)^{\frac{9}{5}} + \frac{1}{11} \right),$$

$$\simeq 0.0347 < 1.$$

Therefore, (8) is satisfied.

The problem (20) has to have a solution $\tilde{\xi}_3 \in \Xi_2$ according to Theorem (3.1).

As a result, by definition (3.1), the boundary value problem (16) has a solution:

$$\xi(s) = \begin{cases} \xi_1(s), & s \in \Upsilon_1, \\ \xi_2(s) = \begin{cases} 0, & s \in \Upsilon_1, \\ \tilde{\xi}_2(s), & s \in \Upsilon_2, \end{cases} \\ \xi_3(s) = \begin{cases} 0, & s \in \Upsilon_1, \\ 0, & s \in \Upsilon_2, \\ \tilde{\xi}_3(s), & s \in \Upsilon_3. \end{cases} \end{cases}$$

In those figures, we present the plot of the function $h(s, \pi, z)$ in the two interval $[0, 1]$, $[1, 2]$ and $[2, 3]$,

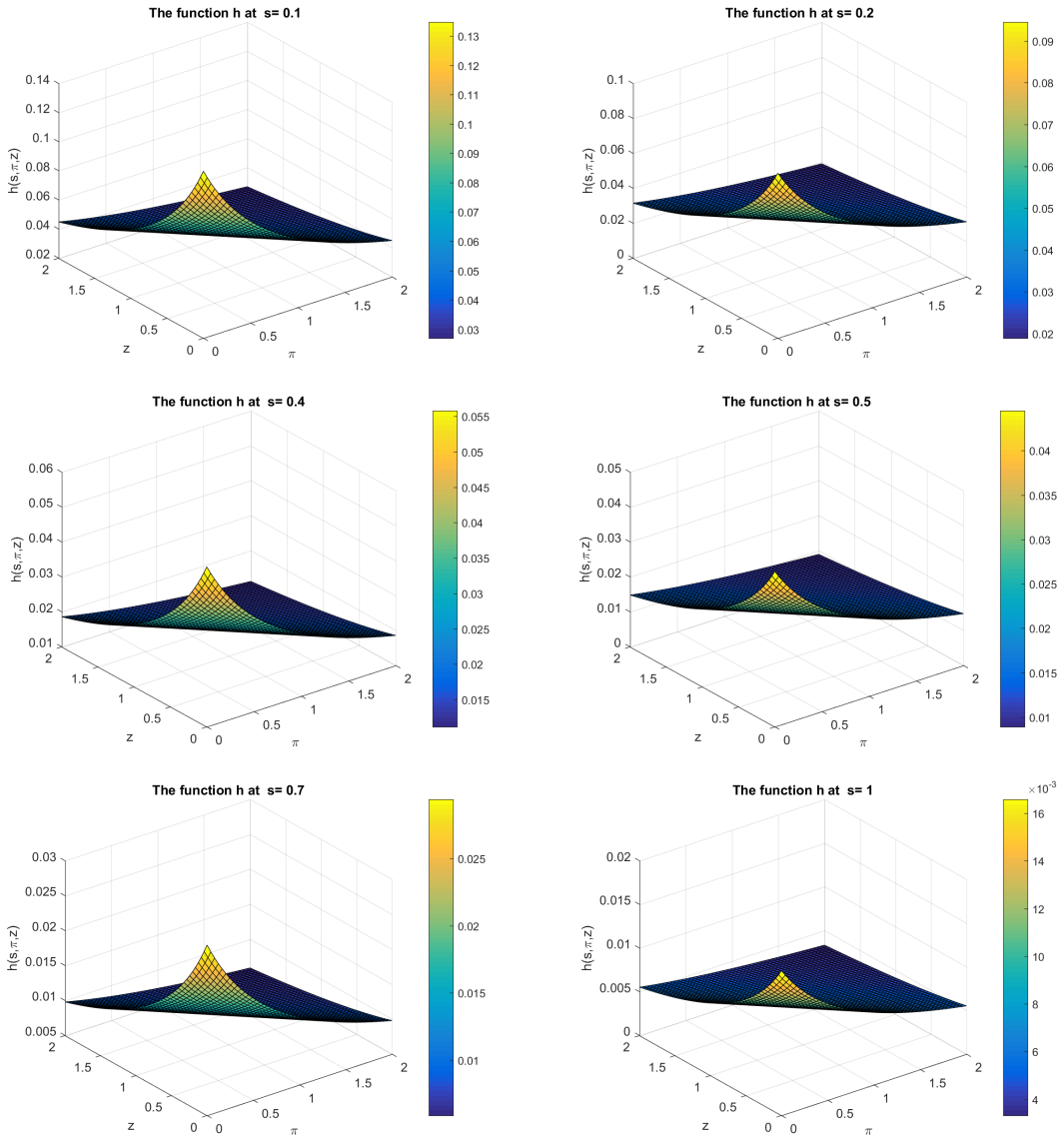


Figure 4: A plot of $h(s, \pi, z)$ for different values of $s \in [0, 1]$.

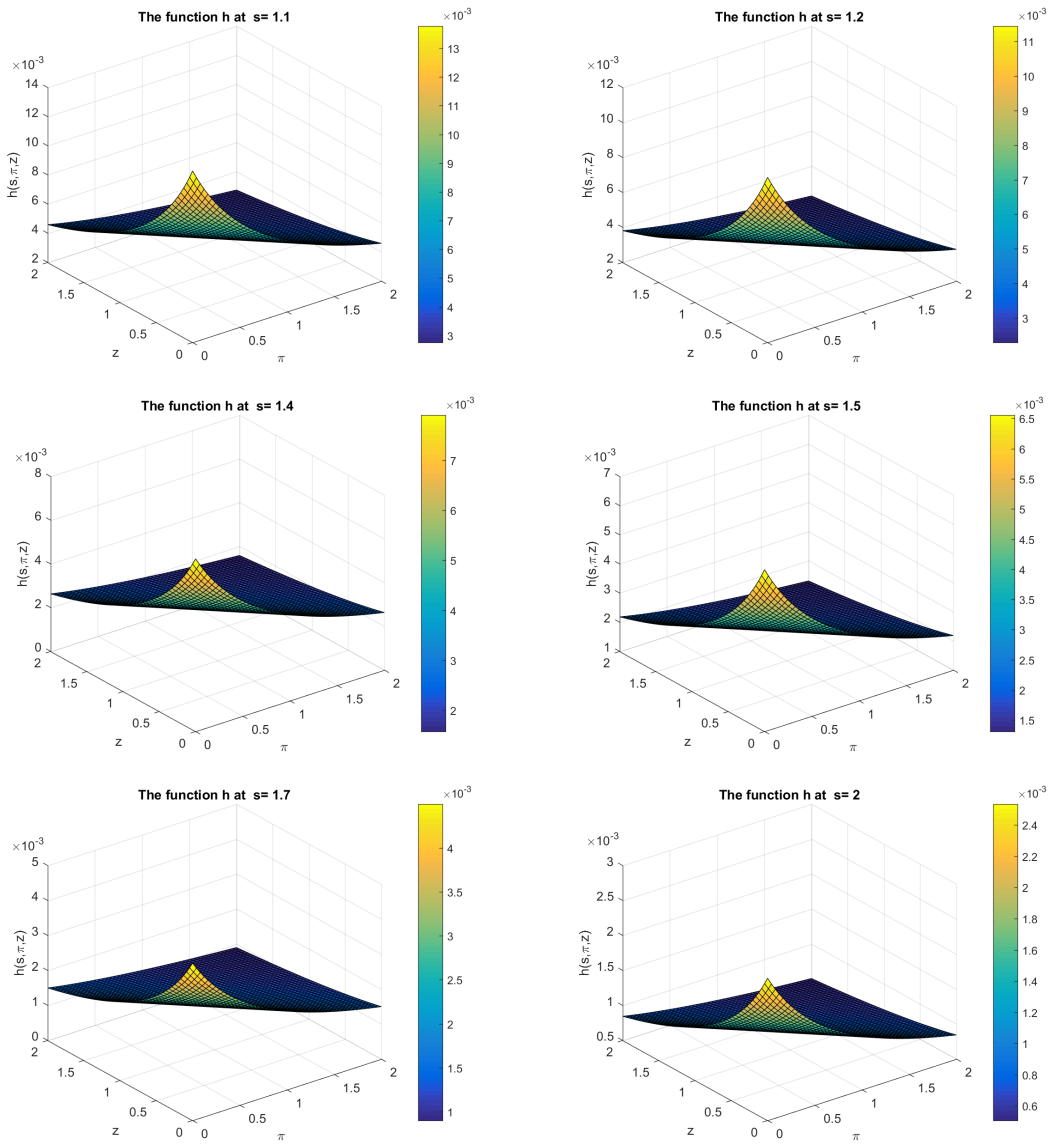


Figure 5: A plot of $h(s, \pi, z)$ for different values of $s \in]1, 2]$.

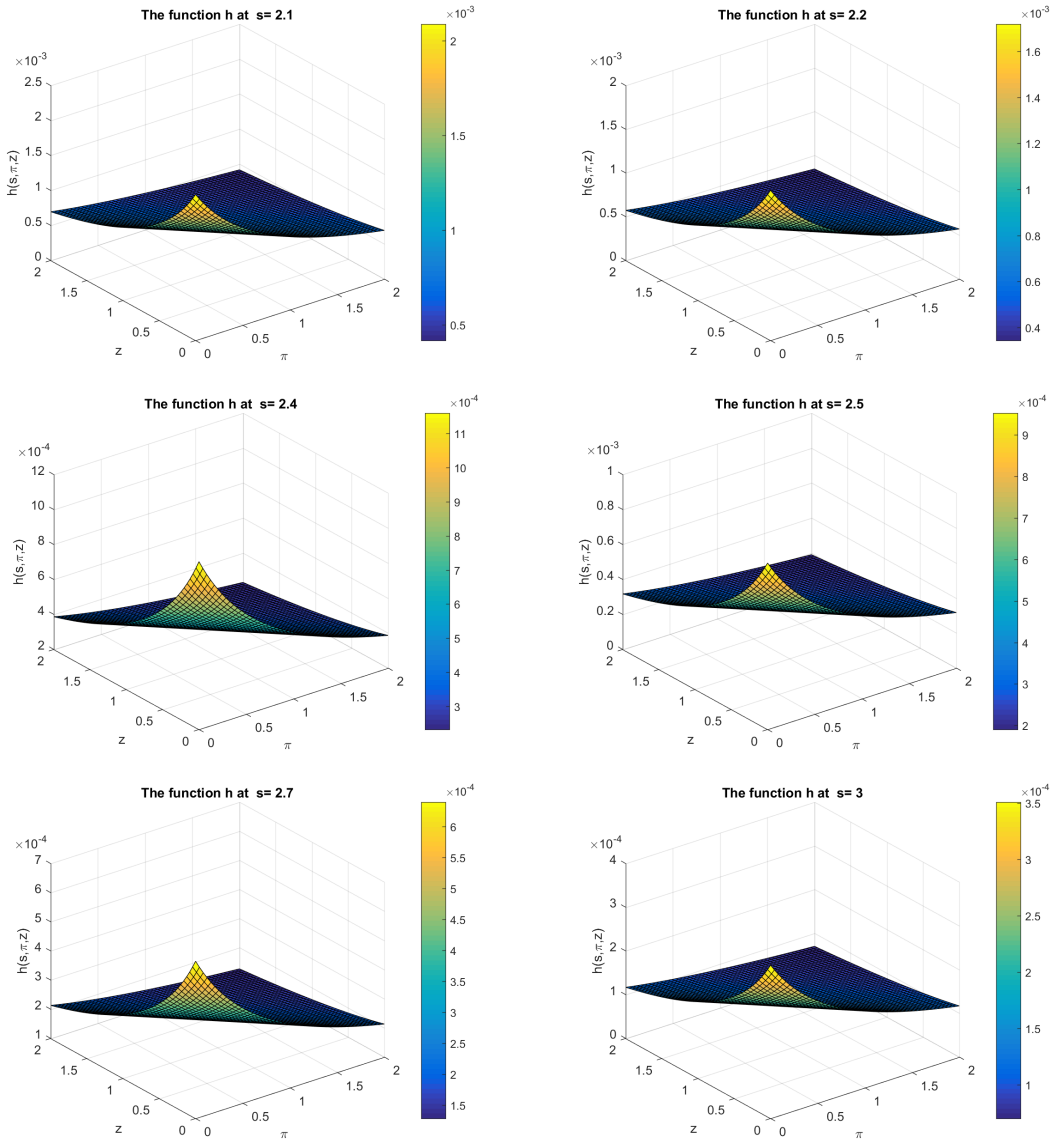


Figure 6: A plot of $h(s, \pi, z)$ for different values of $s \in [2, 3]$.

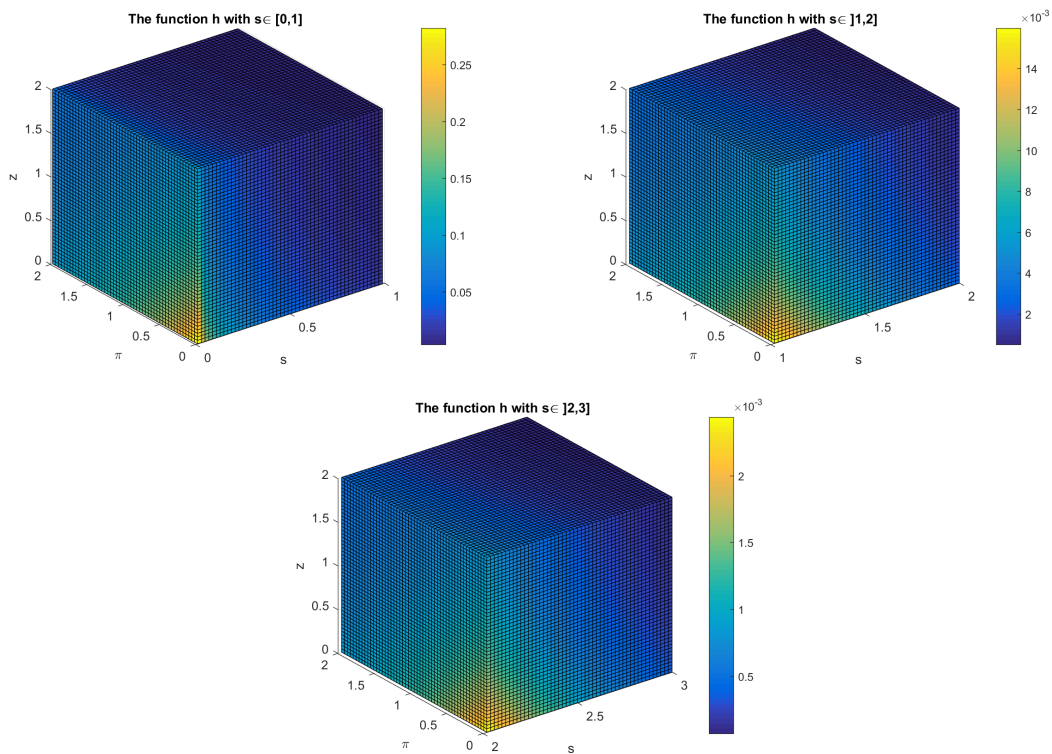


Figure 7: The 3D plot of $h(s, \pi, z)$ for $s \in [0, 1], [1, 2]$ and $[2, 3]$.

In this table, we present the value of $\max_{\pi, z} h(s, \pi, z)$ for different values of $s \in [0, 1], [1, 2]$ and $[2, 3]$. We have made the same observation as in Example 5.1.

$s \in [0, 1]$	0.1000	0.2000	0.3000	0.4000	0.5000	0.7000	1.0000
$\max_{\pi, z} h(s, \pi, z)$	0.1347	0.0945	0.0714	0.0558	0.0445	0.0294	0.0166
$s \in]1, 2]$	1.1000	1.2000	1.3000	1.4000	1.5000	1.7000	2.0000
$\max_{\pi, z} h(s, \pi, z)$	0.0138	0.0114	0.0095	0.0079	0.0066	0.0045	0.0025
$s \in]2, 3]$	2.1000	2.2000	2.3000	2.4000	2.5000	2.7000	3.0000
$\max_{\pi, z} h(s, \pi, z)$	0.0021	0.0017	0.0014	0.0012	0.0010	0.0006	0.0004

6 Conclusions

This research introduced a boundary value problem of Riemann-Liouville fractional differential equations of variable order $\varphi(s)$, in which $\varphi(s)$ is a piecewise constant function. The analytical

solutions have been successfully investigated via three strategies: the DFPTH, Kuratowski's MNK and HUR stability concept. Finally, we illustrated the theoretical results by some numerical examples. Therefore, the results showcased in this paper exhibit immense promise for utilization in diverse applications of multidisciplinary sciences.

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Conflicts of Interest We hereby to declare that there is no conflict of interest among us in conducting this research.

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